PROJECTED WRITTEN NOTES FROM THE M 408D LECTURE
ON THURSDAY, JANUARY 25, 2024, ON
PARTIAL FRACTION DECOMPOSITIONS and an
INTRODUCTION TO Improper Integrals.

CLASS #4

An Integral that is sometimes useful:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

Ex: $\displaystyle\int \frac{1}{x^2+5}\,dx = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$

You can derive this by writing

$$\int \frac{1}{x^2+a^2}\,dx = \int \frac{1}{\left(\sqrt{x^2+a^2}\right)^2}\,dx$$

and use Trig Substitution.

PARTIAL FRACTION DECOMPOSITIONS (PFDs)

Last Time:

$$\frac{P(x)}{Q(x)} = \frac{2x-17}{x^2-5x+4} \overset{\text{(WORK)}}{=} \cdots = \underbrace{\frac{5}{x-1} + \frac{-3}{x-4}}_{\text{The PFD}}$$

$$\int \frac{2x-17}{x^2-5x+4}\,dx = 5\int \frac{1}{x-1}\,dx - 3\int \frac{1}{x-4}\,dx$$

$$= \cdots = \ln\left(|x-1|^5\right) - \ln\left(|x-4|^3\right) + C$$

TO FIND the BASIC FORM of PFD for $\frac{P(x)}{Q(x)}$,

FACTOR THE DENOMINATOR $Q(x)$ as far as possible.

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{\boxed{} \times \boxed{} \times \cdots \boxed{}}$$

These factors cannot be factored further.

SIDE NOTE:

If $\deg P(x) \geq \deg Q(x)$, divide $P(x)$ by $Q(x)$ longhand and work with $\frac{\text{Remainder}}{Q(x)}$.

Ex: $\frac{P(x)}{Q(x)} = \frac{x^3-1}{x^2+1}$. Do this:

$$\begin{array}{r} x \\ x^2+0x+1 \overline{\big) x^3+0x^2+0x-1} \\ \cancel{+}\,x^3 \cancel{+} 0x^2 \cancel{+} x \\ \hline -x-1 \end{array}$$

$$\frac{x^3-1}{x^2+1} = x + \frac{-x-1}{x^2+1}$$ ← Get the PFD of this.

Finding the Partial Fraction Decomposition (PFD) of $\frac{P(x)}{Q(x)}$

when the deg P(x) < deg Q(x) :

1) **Identify the <u>Basic Form</u> of the PFD (factor Q(x) as far as possible)**

2) **Write Equation:** $\frac{P(x)}{Q(x)} =$ **Basic Form**

3) **Multiply by Q(x) and solve for the unknowns A, B, C, etc.**

Example: **Let** $\frac{P(x)}{Q(x)} = \frac{2x-17}{x^2-5x+4} = \frac{2x-17}{(x-1)(x-4)}$

<u>Basic Form</u>

$$\frac{P(x)}{Q(x)} = \frac{2x-17}{(x-1)(x-4)} = \frac{A}{(x-1)} + \frac{B}{(x-4)} \longleftarrow$$

$$\mathbf{Q(x)}\left(\text{------------} \right) = \left(\text{------------} \right)\mathbf{Q(x)}$$

$$(x-1)(x-4)\left(\frac{2x-17}{(x-1)(x-4)} \right) = \left(\frac{A}{(x-1)} + \frac{B}{(x-4)} \right)(x-1)(x-4)$$

Solve for A and B in : $2x-17 = A(x-4) + B(x-1)$

Set x = 1 : $2-17 = A(1-4) + B(1-1) \rightarrow -15 = -3A \rightarrow A = 5$

Set x = 4 : $8-17 = A(4-4) + B(4-1) \rightarrow -9 = 3B \rightarrow B = -3$

The P. F. D.

$$\frac{2x-17}{(x-1)(x-4)} = \frac{5}{(x-1)} + \frac{-3}{(x-4)} \longleftarrow$$

Basic Forms of the Partial Fraction Decomposition (PFD)

I. When the denominator has the form:

(x – a) (x – b) (x – c) ··· (x – d) ← **Distinct Linear Factors**

Form: **PFD** $= \dfrac{A}{x-a} + \dfrac{B}{x-b} + \cdots + \dfrac{D}{x-d}$

II. When the denominator has the form:

$$(x-a)^k$$

Form: **PFD** $= \dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \dfrac{A_3}{(x-a)^3} + \cdots + \dfrac{A_k}{(x-a)^k}$

III. When the denominator has distinct non-factorable non-repeated quadratic factors of the form $x^2 + bx + c$:

Form: **Each such factor requires a term of the form:** $\dfrac{Ax+B}{x^2+bx+c}$

IV. When the denominator is a power of a non-factorable quadratic factor $(x^2 + bx + c)^k$:

Form: **PFD** $= \dfrac{A_1x+B_1}{x^2+bx+c} + \dfrac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(x^2+bx+c)^k}$

V. When the denominator factors completely into a combination of the above, these basic Forms string together to make the basic Form of the whole.

Consider $\dfrac{4x+3}{x^2-3x-10} = \dfrac{P(x)}{Q(x)}$, $Q(x) = (x-5)(x+2)$

A Type I BASIC FORM EXAMPLE

$$\frac{4x+3}{(x-5)(x+2)} = \frac{A}{(x-5)} + \frac{B}{(x+2)}$$

FIND A and B:

$$4x+3 = \left(\frac{A}{x-5} + \frac{B}{x+2}\right)(x-5)(x+2)$$

(*) $4x+3 = A(x+2) + B(x-5)$

⋮ After some work solving for A and B, we get:

$$A = \frac{23}{7}, \quad B = \frac{5}{7}$$

$$\frac{4x+3}{(x-5)(x+2)} = \frac{\frac{23}{7}}{x-5} + \frac{\frac{5}{7}}{x+2}$$ ← the PFD

A Type II Basic Form Example

$$\frac{3x}{(x-3)^2} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2}$$

$$(x-3)^2\left(\frac{3x}{(x-3)^2}\right) = \left(\frac{A}{(x-3)} + \frac{B}{(x-3)^2}\right)(x-3)^2$$

$$(*) \quad 3x = A(x-3) + B$$

$$\leftarrow Ax - 3A + B$$

method 1

$$3x + 0 = Ax + (-3A + B)$$

Equate the coefficients of the terms with the same power of x:

$$\underline{A = 3} \qquad -3A + B = 0$$

$$(-3)(3) + B = 0$$

$$\underline{B = 9}$$

method 2

$$3x = A(x-3) + B \quad (*)$$

Set $x = 3$.

$$9 = A(0) + B \Rightarrow \underline{B = 9}$$

Set $x = 4$ (by choice)

$$12 = A(1) + 9$$

$$\underline{A = 3}$$

$$\frac{3x}{(x-3)^2} = \frac{3}{(x-3)} + \frac{9}{(x-3)^2}$$

$\leftarrow$ The PFD

A TYPE III BASIC FORM EXAMPLE

(Ex 5 on p 511)

$$\frac{P(x)}{Q(x)} = \frac{2x^2 - x + 4}{x^3 + 4x} = ? \text{ as a PFD.}$$

$$x^3 + 4x = x(x^2 + 4) = (x - 0)(x^2 + 4)$$

Can x^2+4 be factored further?
Here is how to know:

$$x^2 + 4 = 0$$

$$x^2 = -4, \quad \text{No Solution!}$$

So, x^2+4 cannot be factored further.

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Cx + D}{(x^2+4)}$$

$$(x)(x^2+4)\left(\frac{2x^2 - x + 4}{x(x^2+4)}\right) = \left(\frac{A}{x} + \frac{Cx + D}{x^2+4}\right)(x)(x^2+4)$$

$$2x^2 - x + 4 = A(x^2+4) + (Cx + D)(x)$$

$$2x^2 - x + 4 = Ax^2 + 4A + Cx^2 + Dx$$

$$2x^2 - x + 4 = (A + C)x^2 + Dx + 4A$$

$$2x^2 + (-1)x + 4 =$$

$A + C = 2$, $\underline{D = -1}$, $4A = 4$

$\underline{A = 1}$

$1 + C = 2$

$\underline{C = 1}$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{1}{x} + \frac{x-1}{x^2+4}$$

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4} \right) dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

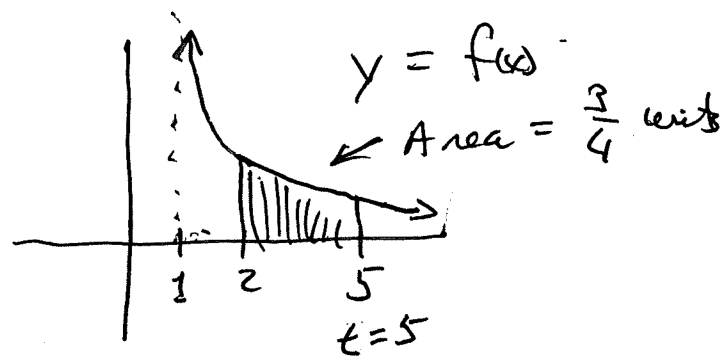
Improper Integrals (Sec 7.8)

Review: The Definite Integral of f over $[a,b]$ is

$$\int_a^b f(x)\,dx = \text{The limit of Riemann Sums over the } \underline{\text{closed interval } [a,b]}$$

Ex: Consider $f(x) = \dfrac{1}{(x-1)^2}$ for $x > 1$, i.e. on $(1,\infty)$

$$\int_2^5 \frac{1}{(x-1)^2}\,dx = \frac{3}{4}$$



$t = 5$

$t \to \infty$

Let t increase without bound.

AN Improper Integral

Define $\int_2^\infty \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{(x-1)^2}\,dx$

If this limit exist, we say
"The Improper Integral is <u>Convergent</u>"
and the value of the limit.

If not, we say
"The Improper Integral Diverges".

$$\int_2^\infty \frac{1}{(x-1)^2} = \lim_{t\to\infty} \int_2^t \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty} \int_1^{t-1} \frac{1}{u^2}\,du$$

$\int \frac{1}{u^2}\,du = \int u^{-2}\,du$

$-\frac{1}{u} + C$

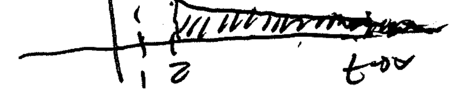
let $u = x-1$
$du = dx$
when $x = 2$, $u = 1$
when $x = t = t-1$

$$= \lim_{t\to\infty} \left(-\frac{1}{u}\right)\Big|_1^{t-1}$$

$$= \lim_{t\to\infty} \left(\left(-\frac{1}{t-1}\right) - \left(-\frac{1}{1}\right)\right)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{\underbrace{(t-1)}_{\to\infty}} + 1\right) = 0 + 1 = 1$$

The Improper Integral $\int_2^\infty \frac{1}{(x-1)^2}\,dx$ is Convergent and has the value 1.



IMPROPER INTEGRALS

Type I : [OVER AN UNBOUNDED INTERVAL]

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$ When the limit exists!

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$ When the limit exists!

When the limit exists for both and with $a=b$; Then $\rightarrow$ $$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^{\infty} f(x)\,dx + \int_{-\infty}^{a} f(x)\,dx .$$

Type II : [When function $f(x)$ fails to be continuous
A: at the left endpoint a
B: at the right endpoint b
C: at some number c with $a<c<b$]

A: $$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$ When limit exists.

B: $$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$ When limit exists.

C: $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t\to c^-} \int_a^t f(x)\,dx + \lim_{t\to c^+} \int_t^b f(x)\,dx$$

When these limits exist.

Ex: